Phase transitions in an aging network

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We consider a growing network in which an incoming node gets attached to the ith existing node with the probability $\Pi_i \propto k_i{}^{\beta}\tau_i^{\alpha}$, where k_i is the degree of the ith node and τ_i its present age. The phase diagram in the $\alpha-\beta$ plane is obtained. The network shows scale-free behavior, i.e., the degree distribution $P(k) \sim k^{-\gamma}$ with $\gamma=3$ only along a line in this plane. Small world property, on the other hand, exists over a large region in the phase diagram.

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Complex weblike structures describe a wide variety of systems of high technological and intellectual importance. The statistical properties of many such networks have been studied recently with much interest. Such networks with complex topology are common in nature and examples include the World Wide Web, the Internet structure, social networks, communication networks, and neural networks, to name a few [1–3].

Some of the common features which are of importance in these networks of diverse nature are:

(i) *Diameter of the network*. This is defined as the maximal shortest path over all vertex pairs. The networks in which the diameter ($\mathcal{D}$) varies as the logarithm of the number of nodes (N), i.e, $\mathcal{D} \propto \ln(N)$, are said to have the *small world* (SW) property. On the other hand, if $\mathcal{D}$ varies as a power of N we get what can be termed as *regular networks*. One can also study D, the average shortest distance between pairs of nodes. In general, D and $\mathcal{D}$ have the same scaling behavior.

(ii) *Clustering coefficient*. A common property of many real networks is the tendency to form clusters or triangles, quantified by the *clustering coefficient*. This is in contrast to random networks [4] where small world property is present but the clustering tendency is absent.

(iii) *Degree distribution*. The node degree distribution function $P(k)$ gives the probability that a randomly selected node has exactly k edges. In a random network this is a Poisson distribution. In many real world networks $P(k)$ shows a power law decay and such networks are termed *scale free networks*.

In order to emulate the different features of real networks several models have been proposed. While the Watts-Strogatz [5] network provides an appropriate model for the small world network (i.e., small diameter and finite clustering coefficient), scale free properties of a network can be reproduced by models proposed later by Barabási and Albert (BA) [6] and independently by Hubermann and Adamic [7].

In the BA model, a network is grown from a few nodes and new nodes are added one by one. At a time t, the incoming node is connected randomly to the ith existing node with the attachment probability $\Pi_i(t)$ given by

$$\Pi_i(t) \sim k_i(t), \tag{1}$$

where k_i is the degree of the ith node. The degree distribution in the BA model shows the scale-free behavior

$$P(k) \sim k^{-\gamma}, \tag{2}$$

with $\gamma=3$.

Following its introduction, several modifications in the BA model have been studied. A few of them are worth mentioning here in the context of the present paper. Nonlinear dependence of the attachment probability on k, in the model designed by Krapivsky *et al.* (KRL) [8], shows that the scale-free property exists only for the linear dependence. This nonlinear model has been studied in much detail very recently in Ref. [9]. On the other hand, the BA model on a Euclidean network [10,11] has also been considered in which the attachment probability has been modified as follows:

$$\Pi_i(t) \sim k_i(t)^{\beta} l^{\delta}, \tag{3}$$

where l is the Euclidean length between the new and old nodes. A phase diagram in the $\beta-\delta$ plane was obtained along with other interesting features.

Another important modification in the BA model has been made by incorporating time dependence in the network [12,13]. In real life networks, a time factor may also modulate the attachment probability. In most of the real world networks, aging of the nodes usually takes place, e.g., one rarely cites old papers, or in social networks, people of the same age are more likely to be linked. Dorogovtsev and Mendes (DM) [12] studied the case when the connection probability of the new site with an old one is not only proportional to the degree k but also to the power of its present age τ, such that

$$\Pi_i(t) \sim k_i(t) \tau_i^{\alpha}, \tag{4}$$

and they showed both numerically and analytically that the scale free nature disappears when $\alpha < -1$ (it may be noted here that τ_i is also the "age difference" between the ith node and the new node). Here α governs the dependence of the attachment probability on the age difference of two nodes, i.e., for negative values of α, a new node will tend to attach itself to the younger nodes. Therefore for the extreme case $\alpha \to -\infty$, a new node will attach itself to its immediate predecessor while for the case $\alpha \to \infty$, the oldest and a few very

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old nodes will get more edges. The time dependence presents a competing effect when $\alpha < 0$ but for $\alpha > 0$, the older nodes get even more rich, similar to the rich gets richer effect.

Encouraged by the existence of the various phase transitions observed in the modified BA models, we have further generalized the time dependent BA network. Here we generate a network such that the attachment probability is given by

$$\Pi_i(t) \sim k_i(t)^{\beta}\tau_i^{\alpha}. \tag{5}$$

We expect here that $\beta \neq 1$ will change the behavior of the DM model as in Ref. [8]. The competing effect of α is able to destroy the scale free nature of the DM model ($\beta=1$). The effect of a positive $\beta(>1)$ and negative α could re-instate the scale free behavior as in Ref. [11] and it is also possible to obtain a phase diagram in the $\alpha-\beta$ plane. Formally Eq. (5) is analogous to Eq. (3). However, here the nodes are chronological, i.e., the age of the initial node is t at time t, that of the second node $t-1$, and so on. In the Euclidean network, on the other hand, the coordinates of the nodes are uncorrelated. Moreover, the dimensionality plays an important factor in it.

The known limits of this model are therefore;

(i) $\beta=1$, $\alpha=0$: BA network;

(ii) β any value, $\alpha=0$: KRL network;

(iii) $\beta=1$, α any value: DM network.

When α and β are both zero, we get a random growing network which shows an exponential decay of $P(k)$ [14].

The network is generated in the usual manner where we start with a single node and at every time step the new node gets attached to one of the existing nodes with an attachment probability given by Eq. (5).

We have considered nodes with a single incoming link such that there are no loops and the clustering coefficient is zero. Thus we focus our attention on the degree distribution and the average distance to study the small world and scale free behavior.

From Eq. (5) we predict that for any β as $\alpha \to +\infty$ a *gel* formation is expected when majority of the nodes tend to get attached to a particular node. On the other hand, when $\alpha \to -\infty$ we expect a *regular chain* formation (in the time space) when each node gets attached to its immediate predecessor. The average shortest distance (D) for both $\alpha \to +\infty$ and $\alpha \to -\infty$ is easy to calculate. When $\alpha \to -\infty$, D is given by

$$D = \frac{\sum_{k=1}^{N} [k(k-1) + (N-k)(N-k+1)]}{2N(N-1)} = (N+1)/3.$$

On the other hand, for large values of α, D has a finite value $\sim O(1)$. Hence it is natural to expect a transition from a small world behavior to a regular behavior as α is varied. In fact for $\beta=0$, one can locate approximately the transition point using some simple arguments.

In analogy with Ref. [15], one can define here an "age difference factor" $\Delta\tau_{ij} = |\tau_i - \tau_j|$ if the ith node of age t_i and jth node of age t_j are connected. If the network has been evolved until a time $t(\geqslant 2)$, then for the incoming node we can write

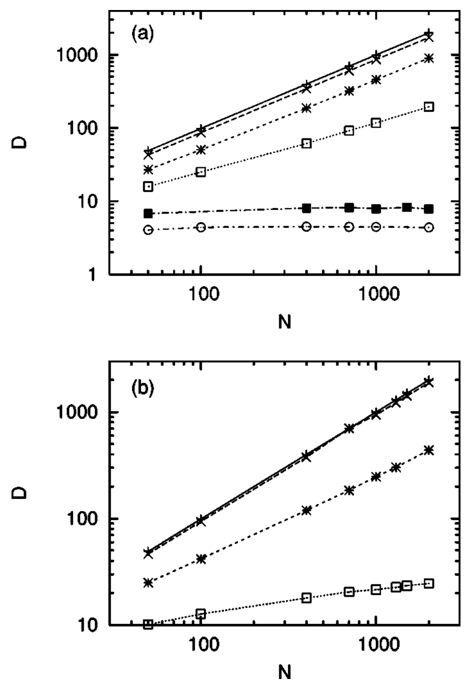

FIG. 1. The variation of the average shortest distance D with N for various values of α at (a) $\beta=2$ and (b) $\beta=0.5$. In (a) the exponent λ changes value from 1 to 0 sharply as we go from the top to the bottom of the figure, corresponding to $\alpha=-10,-5,-3,-2,-1,0$. In (b) λ changes from 1 to a relatively small value as $\alpha=-10,-5,-2,0$ from top to bottom.

$$\langle \Delta\tau \rangle_t = \frac{\int_1^t \tau^{\alpha+1} d\tau}{\int_1^t \tau^{\alpha} d\tau}. \tag{6}$$

For the small world property, the behavior of $\langle \Delta\tau \rangle$ for large t should be studied. From Eq. (6), for large t, $\Delta\tau \sim O(1)$ for $\alpha < -2$ and therefore there can be no small world behavior for $\alpha < -2$ for large networks. On the other hand, for $\alpha > -1$, $\langle \Delta\tau \rangle \sim O(t)$, from which one can expect SW property for $\alpha > -1$. We in fact find a small world to regular network transition at $\alpha = -1$ numerically.

The simulations have been made using a maximum of 2000 nodes for studying small world properties and 4000 nodes for determining degree distribution, using a maximum of 1000 configurations. In the analysis of the small world characteristics, we calculate D for the networks for different values of β and α. The D vs N curve is generally of the form $D \sim N^{\lambda}$,where the exponent λ depends on α (see Fig. 1).

In order to locate the transition to the small world (where λ is either zero or has a very small value) we note the variation of λ with α for different values of β. We observe that for all values of β, there is a sharp fall in λ from unity to a very small value indicating a transition from regular to small world behavior. The transition point shifts to a more negative

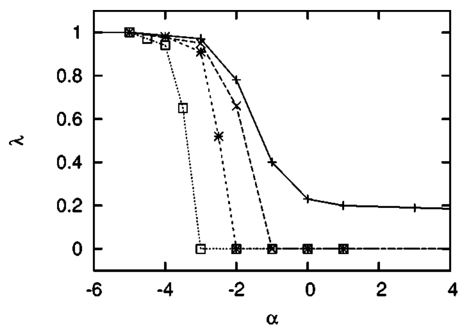

FIG. 2. The variation of the exponent λ with α for different values of β (from right to left, $\beta=0,2,3,4$). There is a sharp transitionin the value of λ from +1 to 0 as network behavior changes from regular to small world.

value of α as we move from smaller to larger values of β. Typical λ vs α plots are shown in Fig. 2.

Next we study the degree distribution $P(k)$ for the network for several values of α and β. For the regular chain limit ($\alpha \to -\infty$), most of the vertices have degree 2, while for the gel phase ($\alpha \to +\infty$), there will be a very high maximal degree and many leaves (i.e., nodes with degree =1). Thus the different phases can be identified from the behavior of $P(k)$. First let us discuss the known case for $\beta=1$.We find that $P(k)$ has an exponential decay at $\alpha=-1$ as in Ref. [12] and has scale-free (SF) behavior for $\alpha > -0.5$. The latter value does not agree with Ref. [12] and the possible reasons of discrepancy are discussed later. For other values of α, $P(k)$ has a stretched exponential (SE) behavior, i.e.,

$$P(k) \sim \exp(-ax^b), \tag{7}$$

where b depends on α. Allowing β to assume values greater than unity, we find that SF behavior exists only for a specific value of $\alpha=\alpha^*$, e.g., at $\beta=3$ we obtain this behavior at $\alpha=\alpha^*=-2.5$ (Fig. 3).

For $\alpha > \alpha^*$, we get a gel-like behavior, while for $\alpha < \alpha^*$, we again get a stretched exponential behavior.The scale-free behavior for $\beta \geqslant 1$ always occurs with $\gamma=3$ as in the BA network. For $\beta < 1$, SF behavior is not observed for any value of α. Here $P(k)$ shows a SE behavior as in Eq. (7).

In Fig. 4, we have shown the phase diagram in the $\alpha-\beta$ plane. We have plotted the phase boundary between the SW and the regular network regions, the curve along which scale-free behavior exists and the line along which $b=1$. The $b=1$ line is not a phase boundary, but it has the interesting property that it has the behavior of a random growing network albeit with nonzero values of α and β. For negative values of α, aging can be regarded as a competitive phenomenon to preferential attachment to the extent that one recovers the random behavior even for high values of β along this line.

A small world network is not necessarily scale free but a scale-free network is usually a small world barring some exceptional or artificial cases (e.g., if one considers N number of BA networks in a series, it is a scale-free but not a small world network). To the right of the scale-free line and above it, the network shows a gel-like behavior. Both the

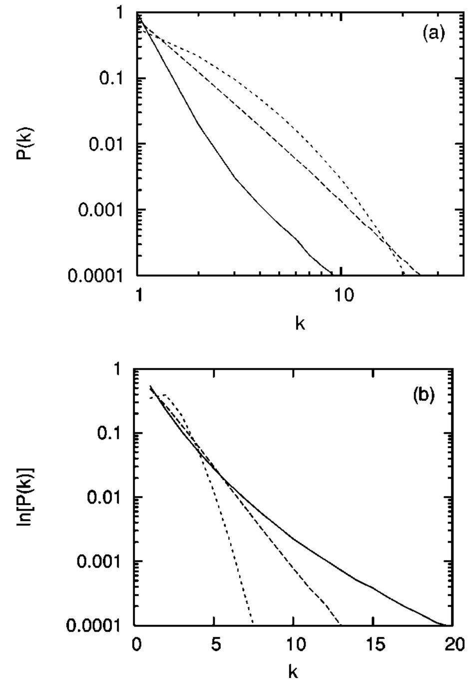

FIG. 3. (a) The $P(k)$ vs k plot in log-log scale for $\beta=3$. Here, at $\alpha=\alpha^*=-2.5$ (dashed line), there is scale free behavior, while for other values (e.g., $\alpha=-2.8,-2.3$) this behaviour is lost. (b) $P(k)$ vs k plot in log-linear scale for $\beta=0.8$. Here exponential behavior is observed at $\alpha=-0.9$ (dashed line), while for other values (e.g., $\alpha=-0.5,-2.0$) stretched exponential nature is observed.

scale-free and gel regions do have the small world property, as expected, but there are finite regions in the phase diagram where the degree distribution is not a power law but of different types (e.g., exponential or stretched exponential) with small world behavior.

In summary, we have generalized the BA network to include time-dependent or aging effects in the attachment

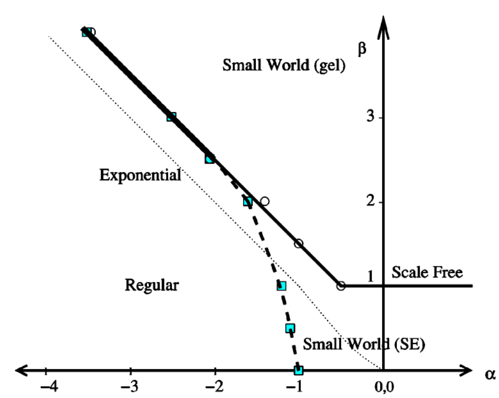

FIG. 4. The phase diagram for the given network in the $\alpha-\beta$ plane. The small world (SW) regions with gel-like as well as stretched exponential (SE) behavior, the regular chain region, and the scale free region are indicated. The network is scale free along the thinner solid line while the broken line is the phase boundary for SW-regular transition and these two lines merge along the thicker solid line. The dotted line is the one along which $b=1$, i.e., where the network is random in nature. (All lines are guides to the eye.)

probability [Eq. (5)] such that both the time dependence and the degree dependence can be parametrically tuned. A phase diagram is obtained in the α-β plane, where α,β are the parameters governing the two factors, respectively. We claim that this is the most generalized network where both time dependence and degree dependence are incorporated in the preferential attachment. There is a quantitative disagreement in the transition point at $\beta=1$ as compared to Ref. [12] which may be because of the finite sizes considered here. The time and effort required to locate phase transition points are considerable and a finite size analysis has not been attempted therefore. Other results known earlier, e.g., gel formation beyond $\beta>1$ for $\alpha=0$, exponential decay of $P(k)$ for both $\alpha,\beta=0$, etc. have been recovered in our simulations. Similar to the Euclidean network [10,11], the scale free behavior is found to exist along a single line here. In fact, as regards the scale-free boundary, the present phase diagram is very much similar to that obtained in Ref. [11]. However, here the network belongs to the BA universality class $(\gamma=3)$ along the entire line. Moreover, one can compare the present results with the one dimensional Euclidean network only for which the phase diagram is available. A phase boundary for small world to regular transition has also been obtained. The network may have small world behavior even when the degree distribution is exponential or stretched exponential. Along the $\alpha=0$ line, the network retains the small world behavior, consistent with the results of Ref. [9], where it was found that D assumed a finite value $(\sim\ln N)$ for networks of different sizes for all values of β.

It is worth mentioning here that the limiting forms of the degree distribution, at extreme values of α, are delta functions in nature, but we have restricted our analysis to finite values of α. Also, we find that the phase diagram shows varied features for values of $\alpha<0$ for which the model corresponds to realistic networks like citation, collaboration, or social networks.

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